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$\mathcal{PT}$ symmetry and its spontaneous breakdown in three dimensions

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Abstract

The mechanism of introducing non-Hermiticity to non-central $\mathcal{PT}$-symmetric potentials through both the φ azimuth and θ polar angles is discussed. Generalizing the results of a previous work it is shown that this can be done also through the polar angle part if appropriate potentials, such as the Scarf I or Rosen–Morse I potentials are used in the eigenvalue equation of the polar component. It is shown that the spontaneous breakdown of $\mathcal{PT}$ symmetry can also occur in these non-central potentials. Several potentials are proposed in the azimuthal eigenvalue equation too, where the use of periodic boundary conditions is essential. Possible generalizations of the results are outlined.

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1. Introduction

The introduction of $\mathcal{PT}$-symmetric quantum mechanics [1] initiated intensive studies both in the field of non-Hermitian quantum Hamiltonians and in the classic area of exactly solvable potentials. The renewed interest in non-Hermitian Hamiltonians started with the discussion of unexpected results concerning manifestly complex potentials that had partly or fully real energy spectrum and also eigenstates with indefinite norm. These Hamiltonians were characterized by $\mathcal{PT}$ symmetry, i.e. invariance with respect to the simultaneous action of the $\mathcal{P}$ space and $\mathcal{T}$ time inversion operations. It soon turned out that $\mathcal{PT}$ symmetry is neither necessary, nor a sufficient condition for possessing real energy spectrum, and $\mathcal{PT}$-symmetric quantum mechanics has also recognized a special case of pseudo-Hermiticity [2]. Much effort has been devoted to exploring the Hermitian connection of non-Hermitian theories, which have been known earlier too under various names. A key element of this connection is the construction of a positive definite metric operator that allows the restoration of the probabilistic interpretation of these theories [3].

Activity in the field of exactly solvable potentials had an impact from $\mathcal{PT}$-symmetric quantum mechanics when it turned out that much of the techniques applied to conventional quantum potentials can be used in this area too. The $\mathcal{PT}$-symmetric versions of real potentials have been analysed from various viewpoints e.g. identifying exact solutions with real and complex-energy eigenvalues in various potential classes [4–10] and discussing the transition between the two domains i.e. the mechanism of the spontaneous breakdown of $\mathcal{PT}$ symmetry [11, 12]; combining $\mathcal{PT}$ symmetry with supersymmetric [7, 13] and algebraic [14–16] techniques; determining the normalization constants and pseudo-norm of $\mathcal{PT}$-symmetric potentials [12, 17, 18], etc.

Although most efforts concerned the study of the bound states of one-dimensional non-relativistic problems, recently, various generalizations of this problem have been proposed, and the investigation of systems with scattering solutions [15, 19], periodic structures [20], coupled channels [21], more particles [22] and relativistic wave equations [23] has been started. Relatively little effort has been paid to discussing $\mathcal{PT}$-symmetric potentials in higher dimensions [24–26]. In a recent systematic study we analysed the conditions under which $\mathcal{PT}$-symmetric two- and three-dimensional potentials can be constructed using the separation of the variables in polar and radial parts [27]. Similar treatment of non-central real potentials has been given already [28], however, the $\mathcal{PT}$-symmetry requirement introduces a number of new elements in the formalism. It turned out, for example, that the angular variables play an important role in introducing imaginary potential components and complex-energy solutions in the system. In [27] this was done using the azimuthal component of the non-central potential. Here we generalize the formalism and demonstrate that with suitable choice of the angular potentials, non-Hermiticity can enter through the polar angle component too, and this extends the range of non-central $\mathcal{PT}$-symmetric solvable potentials considerably.

The arrangement of the paper is as follows. In section 2 we introduce the formalism and develop the strategy to obtain exact solutions of non-central $\mathcal{PT}$-symmetric potentials in terms of one-dimensional ones. In section 3 we propose the application of various potentials in the angular eigenvalue equations, and in particular, that of the exactly solvable $\mathcal{PT}$-symmetric Scarf I and Rosen–Morse I potentials in the polar equation. Finally, we summarize the results and present possible further generalizations in section 4.

2. $\mathcal{PT}$-symmetric Hamiltonians in three dimensions

Let us consider the Schrödinger equation in three spatial dimensions and with constant mass

$$\left(\frac{\mathbf{p}^2}{2m} + V(\mathbf{r})\right)\psi(\mathbf{r}) = -\frac{\hbar^2}{2m}\Delta\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r}). \tag{1}$$

Using polar coordinates and choosing the units $2m = \hbar = 1$ equation (1) takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r^2}\cot\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} - V(r,\theta,\varphi)\psi + E\psi = 0. \tag{2}$$

Assuming that the separation of the variables is possible, we search for the solution as

$$\psi(r,\theta,\varphi) = r^{-1}\phi(r)\sin^{-1/2}\omega(\theta)\tau(\varphi), \tag{3}$$

where $r \in [0,\infty)$, $\theta \in [0,\pi]$ and $\varphi \in [0,2\pi]$. Then (2) turns into

$$\phi''\omega\tau + \frac{1}{r^2}\phi\omega''\tau + \frac{1}{r^2\sin^2\theta}\phi\omega\tau'' - \left(V(r,\theta,\varphi) - \frac{1}{4r^2} - \frac{1}{4r^2\sin^2\theta} - E\right)\phi\omega\tau = 0, \tag{4}$$

where prime denotes the derivative with respect to the appropriate variable.

Next let us assume that the $\omega(\theta)$ and $\tau(\varphi)$ functions satisfy the following second-order differential equations:

$$\omega'' = (P(\theta) - p)\omega, \tag{5}$$

$$\tau'' = (K(\varphi) - k)\tau, \tag{6}$$

where $P(\theta)$ and $K(\varphi)$ are some functions of the respective angular variable, while p and k are constants. These equations can be considered as one-dimensional Schrödinger equations defined on a finite domain, however, they can be more general. In the Schrödinger equation the quantum numbers of the energy eigenfunctions are expected to appear in the energy eigenvalues only, while in the present case they might appear in the potential terms too (at least in the case of (5), as we shall see later). Furthermore, the boundary conditions also have to be specified for the present application. $\tau(\varphi)$ has to be defined with periodic boundary conditions $\tau(0) = \tau(2\pi)$ and $\tau'(0) = \tau'(2\pi)$ on the domain $\varphi \in [0, 2\pi]$, while there is no such requirement for $\omega(\theta)$ on the domain $\theta \in [0, \pi]$. Furthermore, the usual requirement of the vanishing of the bound-state wavefunctions at the boundaries is not prescribed here: the solutions can take on finite value there, in which case they remain normalizable on their finite domain of definition. Due to this finite size one may expect that the potential terms $P(\theta)$ and $K(\varphi)$ might be related to the infinite square well potential with various boundary conditions, so the trend of the eigenvalues p and k may be expected to increase as a quadratic function of the respective principal quantum number.

In the next step we define the non-central potential as

$$V(r, \theta, \varphi) = V_0(r) + \frac{K(\varphi)}{r^2 \sin^2\theta} + \frac{P(\theta)}{r^2} + \frac{1}{r^2 \sin^2\theta}\left(\frac{1}{4} - k\right), \tag{7}$$

where $V_0(r)$ is a spherical potential. Then a radial equation of the type

$$-\phi'' + \left[V_0(r) + \frac{1}{r^2}\left(p - \frac{1}{4}\right)\right]\phi - E\phi = 0 \tag{8}$$

is obtained for $\phi(r)$. This is formally identical with a radial Schrödinger equation in which the $l(l+1)$ angular momentum term is replaced by $p - 1/4$, i.e. formally $p = (l + 1/2)^2$.

Next we investigate under which conditions the Hamiltonian (1), and in particular, the non-central potential (7) exhibits $\mathcal{PT}$ symmetry. For this first we observe that in spherical polar coordinates the space reflection operator $\mathcal{P}$ that acts like $\mathcal{P} : \mathbf{r} \to -\mathbf{r}$ can be factorized into two angular terms

$$\mathcal{P} = \mathcal{P}_\theta \mathcal{P}_\varphi, \tag{9}$$

$$\mathcal{P}_\theta \theta = \pi - \theta, \tag{10}$$

$$\mathcal{P}_\varphi \varphi = \varphi + \pi. \tag{11}$$

These angular $\mathcal{P}$ operators differ from the usual space reflection operator defined in one dimension as $\mathcal{P}x = -x$, so this is one more difference with respect to usual $\mathcal{PT}$-symmetric systems that has to be taken into account when dealing with $\mathcal{PT}$-symmetric potentials in three dimensions. Making use of (9), (10) and (11) the $\mathcal{PT}$ transform of (7) is

$$V(r, \theta, \varphi) = V^*(r, \pi - \theta, \varphi + \pi) \tag{12}$$

$$= V_0^*(r) + \frac{K^*(\varphi + \pi)}{r^2 \sin^2\theta} + \frac{P^*(\pi - \theta)}{r^2} + \frac{1}{r^2 \sin^2\theta}\left(\frac{1}{4} - k^*\right). \tag{13}$$

The general conditions for the $\mathcal{PT}$ symmetry of (13) are thus

$$V_0(r) = V_0^*(r), \tag{14}$$

$$P^*(\pi - \theta) = P(\theta), \tag{15}$$

$$K^*(\varphi + \pi) = K(\varphi), \tag{16}$$

$$k^* = k, \tag{17}$$

so the centrally symmetric component of the potential has to be real, while the two angular potentials have to exhibit a kind of $\mathcal{PT}$ symmetry themselves, furthermore, the eigenvalue of the azimuthal equation (6) has to be real. Note that there are no conditions here regarding the eigenvalue p of the polar equation (5). This might also be complex, in which case the radial-like equation (8) has to be solved with formally complex-energy eigenvalues, which may turn the corresponding energy eigenvalues E into complex. Since E are the eigenvalues of the full Hamiltonian (1) too, this implies that the $\mathcal{PT}$ symmetry of the whole system is spontaneously broken.

There are further conditions that have to be considered when solving such a problem: the potential should not depend on the quantum numbers, so a reparametrization of the individual potential terms might be necessary. This is especially so if the formally different angle-dependent terms in (7) turn out to have similar functional form. This can happen for the last two θ-dependent terms in (7). In this case condition (17) can be dropped.

The strategy of solving the Schrödinger equation with a $\mathcal{PT}$-symmetric non-central potential is the following:

- Solve the azimuthal equation (6) with periodic boundary conditions and check whether the last two terms in (7) can be combined. This is possible if $P(\theta)$ in (5) has a term of the type $\sin^{-2}\theta$. In order to have state independent potential, the quantum numbers from (6) are allowed to appear through k.
- Make sure that the combined term is $\mathcal{PT}$-symmetric and free from the principal quantum numbers of the two eigenvalue equations (5) and (6).
- Substitute the eigenvalue p of (5) into (8) and solve it. Depending on whether p is real or complex, the $\mathcal{PT}$ symmetry of the whole Hamiltonian will be intact or spontaneously broken.

Before closing this section we briefly refer to a possible modification of the formalism that leads to a more compact expression of the potential (7). Although it does not influence the results significantly in general, it can be useful in certain situations. Let us modify (5) such that we introduce in it a term that depends on the eigenvalue k of equation (6)

$$\omega'' = (P(\theta) - p)\omega \equiv \left(\widetilde{P}(\theta) + \frac{k}{\sin^2\theta} - p\right)\omega. \tag{18}$$

With this modification (7) is formally simplified to a form which does not contain the constant k explicitly

$$V(r,\theta,\varphi) = V_0(r) + \frac{K(\varphi)}{r^2\sin^2\theta} + \frac{\widetilde{P}(\theta)}{r^2} + \frac{1}{4r^2\sin^2\theta}. \tag{19}$$

The state independence of $V(r,\theta,\varphi)$ is thus achieved at the price of transferring k-dependence into (18). However, when our objective is finding solvable potentials in three dimensions, the closed solutions of (18) also have to be determined. Unfortunately, the range of exactly solvable potentials containing a $\sin^{-2}\theta$ term is rather limited. In fact, considering potentials

that are non-singular within the $\theta \in (0, \pi)$ domain, it is restricted to the Scarf I and Rosen–Morse I potentials (see, e.g., [4]). These examples will be discussed in the following section.

As a further aspect of this modification we note that prescribing the $\mathcal{PT}$ invariance of $V(r, \theta, \varphi)$ leads to the (14) and (16) conditions and the $\widetilde{P}^*(\pi - \theta) = \widetilde{P}(\theta)$ requirement. This is equivalent with the already established conditions, i.e. (15) and (17), of which the latter one can be dropped if $P(\theta)$ is a potential that contains a $\sin^{-2}\theta$ term. In summary, the modification of the formalism by (18) has no effect on the results as long as we are interested in exactly solvable $\mathcal{PT}$-symmetric potentials in three dimensions.

3. Illustrations

Here we proceed following the strategy outlined at the end of the previous section and present examples for various solvable $\mathcal{PT}$-symmetric non-central potentials in three dimensions. We consider only potentials that are state independent. The results concerning the solution of the azimuthal equation (6) can also be used to generate solvable $\mathcal{PT}$-symmetric non-central potentials in two dimensions. We also comment on previous results mentioned in [27] and discuss how they fit into the general scheme as special cases.

3.1. Solving the azimuthal equation (6)

The condition for the $\mathcal{PT}$ symmetry of this problem (16) is different from the usual conditions applied in the case of one-dimensional $\mathcal{PT}$-symmetric potentials, i.e. $V^*(-x) = V(x)$, so some care should be taken in using previous results. Periodic boundary conditions are also essential here, so in the simplest case periodic potentials can be considered. The example presented in [17] demonstrates that the $\mathcal{PT}$-symmetry requirement prescribes *different* periodicity for the real and imaginary potential components in this case: these have to have even and odd number of periods in the $\varphi \in [0, 2\pi]$ domain, respectively.

A natural choice could be the combination of imaginary step potentials on a ring [25, 26]. Due to the periodic boundary conditions the energy eigenvalues of these systems asymptotically go to those of the infinite square well, but the number and arrangement of the steps are also reflected in the local relative position of the levels. The solutions up to 20 or so can be easily determined for these potentials by graphical and analytical methods. It was also found that by increasing non-Hermiticity (i.e. the height of the steps), complex-energy solutions appear at a certain point [26]. With the use of these potentials in (6) therefore at least a finite set of the solutions of non-central three-dimensional $\mathcal{PT}$-symmetric potentials can be handled. It has to be noted that the potential in [25] does not obey (16), because it has an even number of imaginary steps on the ring, nevertheless, the results are instructive to the case considered here too.

Further possibility could be applying $\mathcal{PT}$-symmetric arrangements of Dirac delta potentials [20] or attempting to construct the Lamé-type potentials [29] with $\mathcal{PT}$ symmetry.

As an exactly solvable example one can also apply the $\mathcal{PT}$-symmetric Scarf I [17] or Rosen–Morse I [18] potentials, which are analytically solvable. Due to the requirement (16), however, these potentials have to be defined in two (or even number of) separate domains on the $[0, 2\pi]$ domain, with periodic boundary conditions. One problem with these potentials is that they possess inverse-square-type singularity at the boundaries, which will thus separate the segments from each other by an impenetrable wall. This can be avoided by using parameters that result in a weakly attractive singularity at the boundaries, which would, in principle allow communication between the individual segments. We do not consider these potentials here, rather we apply them later on in the polar equation (5).

The simplest choice is applying the *real* infinite square well as a special $\mathcal{PT}$-symmetric azimuthal potential: with this choice $K(\varphi) = 0$ and the non-central potential depends only on the polar angle θ, so technically this is the easiest way to illustrate the methods for that component of the wavefunctions. The general solutions can be written in terms of the exponential functions $e^{\pm im\varphi}$, but in order to construct $\tau(\varphi)$ functions that are the eigenfunctions of the $\mathcal{P}_\varphi\mathcal{T}$ operator (see (11)) with unit eigenvalue, a special combination of them has to be taken

$$\tau_m(\varphi) = \frac{i^m}{(2\pi)^{1/2}} \cos(m\varphi). \tag{20}$$

These wavefunctions are $\mathcal{PT}$ normalized as $\langle\tau_n|\mathcal{P}_\varphi|\tau_m\rangle = \delta_{nm}(-1)^m$ and the corresponding energy eigenvalues are $k_m = m^2$. Actually, this choice can also be obtained from the Scarf I and Rosen–Morse I potentials discussed above if they are defined on the $[0, 2\pi]$ domain, with vanishing real and imaginary terms.

3.2. Solving the polar equation (5)

In this case, the requirement of $\mathcal{PT}$ symmetry is simpler both because the operation (10) can be interpreted as an ordinary space reflection and also because the boundary conditions need not be periodic. In principle any solvable or other $\mathcal{PT}$-symmetric potential defined on a finite domain can be applied here, nevertheless, those containing a term of the type $\sin^{-2}\theta$ play a special role, because they allow the construction of non-central potentials that do not depend on the quantum numbers, in particular on that originating from the azimuthal equation (6). The simplest case is when $P(\theta)$ contains only this term: then the $\omega(\tau)$ functions are expressed in terms of associated Legendre functions [27], however, non-Hermiticity can enter the potential only through the azimuthal equation (6) in this case. As a possible generalization, one may consider the $\mathcal{PT}$-symmetric Scarf I [17] or Rosen–Morse I [18] potentials, which have an imaginary component too in addition to the $\sin^{-2}\theta$ term. Combining these potentials with the infinite real square well in the azimuthal equation, the angular wavefunctions become $\mathcal{PT}$-symmetric generalizations of the spherical harmonics.

3.2.1. $P(\theta)$ as the $\mathcal{PT}$-symmetric Scarf I potential. Originally this potential was defined [17] on the $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ domain, so a shift of $\theta = x + \frac{\pi}{2}$ is necessary first: this simply reverses the sign of the odd imaginary term in the potential (or replaces α and β with each other) and has no significant effect on the results. Then

$$P(\theta) = \left(\frac{\alpha^2+\beta^2}{2} - \frac{1}{4}\right)\frac{1}{\sin^2\theta} - \frac{\alpha^2-\beta^2}{2}\frac{\cos\theta}{\sin^2\theta}, \tag{21}$$

and

$$p_n = \left(n + \frac{\alpha+\beta+1}{2}\right)^2, \tag{22}$$

while the bound-state solutions can be written in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(\cos\theta)$.

The actual form of (7) is now

$$V(r,\theta,\varphi) = V_0(r) + \frac{1}{r^2\sin^2\theta}\left(\frac{\alpha^2+\beta^2}{2} - k_m\right) + \frac{K(\varphi)}{r^2\sin^2\theta} - \frac{\cos\theta}{r^2\sin^2\theta}\frac{\alpha^2-\beta^2}{2}, \tag{23}$$

where, in the second term one has to cancel the dependence on m by a parameter change

$$\frac{\alpha^2+\beta^2}{2} - k_m \equiv A = \text{const.} \quad \text{real} \tag{24}$$

Since this requirement makes α and β dependent on m and A, the m-independence of the third term in (23) also has to be guaranteed as

$$\alpha^2 - \beta^2 \equiv 2B = \text{const.} \quad \text{imaginary.} \tag{25}$$

From (24) and (25) it follows that

$$\alpha^2 = A + B + k_m \tag{26}$$

$$\beta^2 = A - B + k_m. \tag{27}$$

In what follows we do not indicate the m-dependence of α and β explicitly, except when it is essential.

$\mathcal{PT}$ symmetry requires now

$$K^*(\varphi + \pi) = K(\varphi), \tag{28}$$

$$2A \equiv \alpha^2 + \beta^2 - 2k_m = (\alpha^*)^2 + (\beta^*)^2 - 2k_m^* \equiv 2A^*, \tag{29}$$

$$2B \equiv \alpha^2 - \beta^2 = -(\alpha^*)^2 + (\beta^*)^2 \equiv -2B^*. \tag{30}$$

Condition (30) is the same as in the usual one-dimensional case, while (29) is more general due to the presence of k_m. If, however, k_m is real, then the same condition follows in this case too: $(\alpha^*)^2 = \beta^2$ [17]. (In the simplest case discussed in the previous subsection, $K(\varphi)$ is the infinite square well and $k_m = m^2$, real.) From (22) it is seen that for $\alpha^* = \beta$ and $\alpha^* = -\beta$ the eigenvalue p is real and complex, respectively.

The radial equation (8) becomes similar to an ordinary radial Schrödinger equation

$$-\phi'' + \left(V_0(r) + \frac{l(l+1)}{r^2}\right)\phi - E\phi = 0, \tag{31}$$

where $l = l_{m,n} = n + \frac{1}{2}(\alpha_m + \beta_m)$. Depending on whether $\alpha_m + \beta_m$ is real or complex, the energy eigenvalue $E = E_{jnm}$ will also be real or complex. (Here j is reserved for the principal quantum number labelling the solutions of (31).) $V_0(r)$ can be any real potential defined on the positive r-axis: in the simplest case it can be the harmonic oscillator or the Coulomb potential, in which case complete analytical solution is possible, but it can also be some quasi-exactly [30] or numerically solvable potential.

3.2.2. $P(\theta)$ as the $\mathcal{PT}$-symmetric Rosen–Morse I potential. The solutions of this potential and its comparison with the $\mathcal{PT}$-symmetric Scarf I potential have been discussed recently [18]. A main difference with respect to this latter potential is that in the natural form of the related eigenvalue equation the principal quantum number appears in one of the coordinate-dependent terms, so a parameter transformation similar to that applied in the previous part has to be performed in order to obtain the solutions [4, 31]. Since such a dependence will also appear on the quantum number originating from the azimuthal equation, we start from the original form of the eigenvalue equation. According to this

$$P(\theta) = \left(n + \frac{\alpha+\beta}{2}\right)\left(n + \frac{\alpha+\beta}{2} + 1\right)\frac{1}{\sin^2\theta} + \mathrm{i}\frac{\alpha^2 - \beta^2}{2}\cot\theta \tag{32}$$

and

$$p_n = \left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{\alpha-\beta}{2}\right)^2, \tag{33}$$

and the bound-state solutions are again written in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(-\mathrm{i}\cot\theta)$.

The corresponding non-central potential (7) is

$$V(r,\theta,\varphi) = V_0(r) + \frac{1}{r^2\sin^2\theta}\left[\left(n+\frac{\alpha+\beta+1}{2}\right)^2 - k_m\right] + \frac{K(\varphi)}{r^2\sin^2\theta} + \frac{\mathrm{i}}{r^2}\frac{\alpha^2-\beta^2}{2}\cot\theta. \tag{34}$$

Now the second term depends both on n and m, so in order to make (34) state independent the following parameter change has to be made

$$\left(n+\frac{\alpha+\beta+1}{2}\right)^2 - k_m \equiv A = \text{const.} \quad \text{real} \tag{35}$$

$$\alpha^2-\beta^2 \equiv 2B = \text{const.} \quad \text{real}. \tag{36}$$

Equations (24) and (25) imply that

$$\alpha = (A+k_m)^{1/2} - n - \frac{1}{2} + \frac{B}{2}\left((A+k_m)^{1/2} - n - \frac{1}{2}\right)^{-1}, \tag{37}$$

$$\beta = (A+k_m)^{1/2} - n - \frac{1}{2} - \frac{B}{2}\left((A+k_m)^{1/2} - n - \frac{1}{2}\right)^{-1}. \tag{38}$$

This is formally similar to the results of [18] with the exception that instead of a constant parameter there is an m-dependent one appearing here.

The $\mathcal{PT}$-symmetry requirement implies

$$K^*(\varphi+\pi) = K(\varphi), \tag{39}$$

$$A \equiv \left(n+\frac{\alpha+\beta+1}{2}\right)^2 - k_m = \left(n+\frac{\alpha^*+\beta^*+1}{2}\right)^2 - k_m^* \equiv A^*, \tag{40}$$

$$2B \equiv \alpha^2-\beta^2 = (\alpha^*)^2 - (\beta^*)^2 \equiv 2B^*. \tag{41}$$

These equations are again more general than in the usual case of the $\mathcal{PT}$-symmetric Rosen–Morse potential [18], but it turns out that for real k_m, α and β are also real, so

$$p_{n,m} = \left((A+k_m)^{1/2} - n - \frac{1}{2}\right)^2 + \frac{B^2}{4}\left((A+k_m)^{1/2} - n - \frac{1}{2}\right)^{-2} \tag{42}$$

is also real. (This is the case if again, $K(\varphi)$ is the infinite square well and $k_m = m^2$.) However, for complex values of k_m not only α and β, but $p_{n,m}$ will also become complex. This also means that the nature of the energy eigenvalues of the radial equation (8) and thus those of the whole system depend on whether the eigenvalues of the azimuthal equation (6), k_m are real or complex. The central potential $V_0(r)$ in (8) can be chosen in the same way as before.

4. Summary and outlook

Non-central $\mathcal{PT}$-symmetric quantum potentials have been analyzed in three dimensions by separating the variables in spherical polar coordinates. The original problem was separated into two angular and a radial eigenvalue problem, and the requirement of $\mathcal{PT}$ symmetry was defined for these components separately. The role of the boundary conditions, and in particular, that of the periodic boundary condition characterizing the azimuthal equation has

been discussed. It was shown that although usual one-dimensional potentials can be used to formulate the angular equations, some care has to be taken in adapting these systems to the actual situation.

A three-step strategy was formulated to obtain the solution of these potentials. In the first step the azimuthal equation is solved with periodic boundary conditions. The use of various periodic $\mathcal{PT}$-symmetric potentials has been proposed, including the arrangement of step and Dirac delta potentials, as well as Lamé-type and some well known exactly solvable potentials. The eigenvalue of the azimuthal equation then appears in one of the potential terms of the polar equation, so the change of parameters is required to make the full potential (7) state independent. For this the application of potentials containing the term $\sin^2\theta$ is necessary. In addition to the case of this symmetric real potential, the use of the $\mathcal{PT}$-symmetric Scarf I and Rosen–Morse I potentials has been proposed, as they may possess an imaginary component too. If these potentials are combined with the infinite real square well in the azimuthal equation, the angular component of the wavefunction will become a $\mathcal{PT}$-symmetric generalization of the spherical harmonics.

In the last step the eigenvalues of the polar equation appear in the radial equation formally in a centrifugal-like term, and the solution of the radial equation supplies the energy eigenvalues of the full non-central $\mathcal{PT}$-symmetric potential too. It was shown that this mechanism can lead to the spontaneous breakdown of $\mathcal{PT}$ symmetry if the centrifugal-like term in the radial equation becomes complex. Completely analytical solutions can be derived using radial potentials that are solvable in the conventional setting for arbitrary value of the orbital angular momentum l, e.g. for the harmonic oscillator and Coulomb potentials. It is notable that in one-dimensional $\mathcal{PT}$-symmetric quantum mechanics the Coulomb potential can be discussed only on trajectories outside the real x-axis, because otherwise the boundary conditions cannot be enforced [4, 32].

After setting the general formalism, it seems worthwhile to continue the study of concrete examples. With this, typical aspects of both one-dimensional $\mathcal{PT}$-symmetric and multi-dimensional Hermitian potentials could be investigated. These could include calculating the pseudo-norm, analyzing the possible presence of quasi-parity, the identification of possible degeneracy patterns and symmetry groups related to them, etc. Extending these studies to pseudo-Hermitian systems in $d > 1$ also seems worthwhile.

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